Advance Selling in the Presence of Experienced Consumers *

Oksana Loginova† X. Henry Wang‡ Chenhang Zeng§

June 30, 2011

Abstract

The advance selling strategy is implemented when a firm offers consumers the opportunity to order its product in advance of the regular selling season. Advance selling reduces uncertainty for both the firm and the buyer and enables the firm to update its forecast of future demand. The distinctive feature of the present theoretical study of advance selling is that we divide consumers into two groups, experienced and inexperienced. Experienced consumers know their valuations of the product in advance. The presence of experienced consumers yields new insights. Specifically, pre-orders from experienced consumers lead to a more precise forecast of future demand by the firm. We show that the firm will always adopt advance selling and that the optimal pre-order price may or may not be at a discount to the regular selling price.

Key words: advance selling, the Newsvendor Problem, demand uncertainty, experienced consumers, inexperienced consumers.

* We thank Stanislav Kolenikov for his help on proofs.

† Department of Economics, University of Missouri-Columbia, loginovao@missouri.edu (corresponding author).

‡ Department of Economics, University of Missouri-Columbia, wangx@missouri.edu.

§ Department of Economics, University of Missouri-Columbia, czvd7@mail.mizzou.edu.

1 Introduction

Advance selling occurs when firms and retailers offer consumers the opportunity to order the product or service in advance of the regular selling season. Remarkable developments in the Internet and information technology have made advance selling an economically efficient strategy in many product categories. Examples include new books, movies and CDs, software, electronic games, smart phones, travel services and vacation packages.

There are several major advantages of advance selling. First, advance selling reduces uncertainty for both the firm and the buyer, because advance orders are pre-committed. In situations when the firm needs to decide how much to produce (procure) prior to the regular selling season, advance orders reduce demand uncertainty. For the buyer, an advance order guarantees delivery of the product in the regular selling season, possibly at a discount to the retail price. Second, orders from advance selling may provide valuable information for the firm to better forecast the future demand. In particular, the firm may be able to update its forecast of the size of consumer pool and the distribution of consumers' valuations. Finally, advance selling may increase the overall demand. Indeed, when a consumer pre-orders the product, she commits to purchase it. In the absence of advance selling the same consumer will not purchase the product if she learns her valuation is low.

The motivation for the present study is based on two observations. One is that many pre-orders are from consumers who have previous experience with the product or its earlier versions. The other is that some products were not made available for pre-orders when they were first introduced, but pre-orders became possible for later versions. These observations point to an important role played by experienced consumers in advance selling.

Table 1 reports the product release history of several well-known products. These products are also widely cited as examples of advance selling. In the first four examples there were no pre-orders for the first one or two versions, and pre-orders were offered for later versions, some with discount and some without.

While inexperienced consumers learn more about their valuations of the product when it becomes available, experienced consumers are likely to have a good idea about their valuations of the product in advance. Therefore, experienced consumers have less incentives to wait until the regular selling season. It follows that when there are experienced consumers, advance selling is more likely to be utilized by consumers. In addition, pre-orders from experienced consumers are more informative than those from inexperienced consumers. One can thus conclude that the presence of experienced consumers makes the first two of the aforementioned advantages of advance selling more pronounced.

A number of papers in the literature have emphasized some or all of the three advantages of advance selling (see the literature review in Section 2), but none have modeled experienced consumers and the role they play in advance selling. This paper is the first study of advance selling with both experienced consumers and inexperienced consumers.

Our model has two periods. The first is the advance selling season and the second is the regular selling season. In the first period, the firm chooses whether to make its product available for pre-orders, and if so, the level of discount from the retail price. There are two groups of consumers – experienced and inexperienced. Experienced consumers know their valuations of the product from the outset, while inexperienced consumers learn their valuations only in the

Table 1: Release history and pre-order availability for several products

Product	Version	Release date	Pre-order availability/Discount
Amazon Kindle	Kindle	Nov. 19, 2007	No
	Kindle 2	Feb. 23, 2009	Yes/No discount
	Kindle 3	Aug. 27, 2010	Yes/No discount
Harry Potter	Book 1	Sep. 1, 1998	No
	Book 2	Jun. 2, 1999	No
	Book 3	Sep. 8, 1999	Yes/40% off
	Book 4	Jul. 8, 2000	Yes/40% off
	Book 5	Jun. 21, 2003	Yes/40% off
	Book 6	Jul. 16, 2005	Yes/40% off
	Book 7	Jul. 21, 2007	Yes/49% off
iPhone	iPhone	Jun. 29, 2007	No
	iPhone 3G	Jul. 11, 2008	No
	iPhone 3GS	Jun. 19, 2009	Yes/No discount
	iPhone 4	Jun. 24, 2010	Yes/No discount
iPod Touch	iPod Touch 1st	Sep. 5, 2007	No
	iPod Touch 2nd	Sep. 9, 2008	No
	iPod Touch 3rd	Sep. 9, 2009	Yes/No discount
	iPod Touch 4th	Sep. 8, 2010	Yes/No discount
PlayStation	PlayStation 1	Sep. 9, 1995	Yes/No discount
	PlayStation 2	Oct. 26, 2000	Yes/No discount
	PlayStation 3	Nov. 17, 2006	Yes/No discount
Nintendo Wii	Wii Fit	May 21, 2008	Yes/$20 off
	Wii Fit Plus	Oct. 4, 2009	Yes/$10 off

second period. All consumers decide whether to pre-order the product (if this option is available) or wait until the regular selling season, in which they will face a probability of not being able to get the product (the stock-out probability). At the conclusion of the first period, the firm must choose its production quantity, which has to be at least the size of pre-orders. The product is delivered at the end of the second period.

Consumers are heterogeneous in their valuations, which are assumed to follow a normal distribution. The firm does not know the mean of this distribution. The group size of experienced consumers is fixed and known to the firm. However, the firm is uncertain about the number of inexperienced consumers.

In the second period the firm faces the Newsvendor Problem by analogy with the situation faced by a newsvendor who must decide how many copies of the day's paper to stock on a newsstand before observing demand, knowing that unsold copies will become worthless by the end of the day. If the produced quantity is greater than the realized demand, the firm must dispose of the remaining units at a loss (due to the salvage value being below the marginal cost). If the produced quantity is lower than the realized demand, the firm forgoes some profit.[1]

Our main research questions are the following. Will the firm adopt the advance selling strategy? If so, will an advance selling discount be offered? How do experienced and inexperienced consumers behave in the advance selling season? What can the firm learn from pre-orders? How much should the firm produce? How are the answers to (some of) these questions affected by parameters of the model, such as the salvage value and the composition of experienced/inexperienced consumers in the population?

Our main results are summarized below.

- The firm always adopts advance selling. Advance selling may be at a discount and may be not.

- Experienced consumers never wait until the regular selling season. Inexperienced consumers sometimes pre-order, sometimes wait until the regular selling season. When the pre-order discount is deep, inexperienced consumers pre-order. When the discount is moderate, inexperienced consumers pre-order if the mean of the distribution from which their valuations are drawn is high, and wait if otherwise.

- The firm learns from pre-orders, which softens the Newsvendor Problem. It learns whether there are any consumers who have chosen to wait until the regular selling season. If nobody waits, the firm only needs to fill all pre-orders. In the case when some consumers wait, the firm learns the mean of consumers' valuations. However, the uncertainty about the number of inexperienced consumers remains.

- Our sensitivity analysis in regard to changes in some parameters of the model yields several interesting results, some intuitive and some counterintuitive. For example, as the salvage value decreases, the firm's expected profit may decrease (intuitive), but may also increase (counterintuitive). Likewise, as the proportion of experienced consumers decreases, the firm's expected profit may decrease (intuitive), but may also increase (counterintuitive).

Our paper contains several contributions to the literature on advance selling. First, as mentioned earlier, we are the first to study advance selling in a model with experienced consumers. We believe our model captures an important aspect of the advance selling phenomena. Second, learning by the firm in our model is not only on the consumer pool but also on the distribution of consumers' valuations of the product. Finally, the stock-out probability that consumers face when they wait until the regular selling season is endogenously determined in our model. In the

[1] If there is no uncertainty about the second-period demand or if the salvage value equals the marginal cost, then the Newsvendor Problem disappears.

literature, the stock-out probability has been modeled as exogenously given. We think that the correct way to model the stock-out probability is through endogenous determination, since this probability affects consumers' choices in the advance selling season and these choices in turn affect the stock-out probability in the regular selling season.

After the literature review (Section 2), the rest of the paper is organized as follows. In Section 3 we introduce the model. Section 4 is devoted to equilibrium analysis. In Section 5 sensitivity analysis results are presented. In Section 6 we consider two extensions. Concluding remarks are provided in Section 7. Proofs of all lemmas and propositions, as well as derivations for some expressions and claims, are relegated to Appendix.

2 Literature Review

Several strands of the literature have studied advance selling. One deals with advance selling from manufactures to retailers, e.g. Cachon (2004) and Taylor (2006). Another is on advance selling from firms and retailers to consumers under limited capacity, with applications to the airline and hotel industries (Xie and Shugan, 2001, and Liu and van Ryzin, 2008). The literature that is closest to the present study is on advance selling from firms and retailers to consumers without capacity constraints.

Our review below focuses on the third strand.[2] Two modeling approaches have been adopted by researchers. In the first approach consumers are non-strategic in their decisions on whether to pre-order the product. In the second approach consumers are strategic.

Papers that model consumers as non-strategic include Weng and Parlar (1999), Tang, Rajaram, Alptekinoğlu, and Ou (2004), McCardle, Rajaram, and Tang (2004), and Chen and Parlar (2005). In all of these papers the fraction of consumers who place advance orders is an exogenously given decreasing function of the advance selling price.[3] In Tang, Rajaram, Alptekinoğlu, and Ou (2004) and McCardle, Rajaram, and Tang (2004) there are two brands belonging to rivalry firms. Advance selling by a firm attracts customers of the other brand. The former paper examines the decision on advance selling by a single firm, while the latter focuses on competition between two firms in adopting the advance selling strategy.

Several papers have treated consumers as strategic. Strategic consumers compare the options of ordering in advance and of waiting until the regular selling season. Zhao and Stecke (2010) classify consumers according to whether they are loss averse. A loss averse consumer is more averse to a negative surplus (when the realized valuation is below the advance selling price) than is attracted to the equivalent positive surplus. Prasad, Stecke, and Zhao (2011) divide consumers into two groups. The informed group consists of consumers who know about the option to buy in advance, while the uninformed group is not aware of this option. Chu and Zhang (2011) allow the firm to control the release of information about the product at pre-order.

The common issues present in the literature are (i) the Newsvendor Problem, and (ii) learning and updating by the firm.[4] The Newsvendor Problem arises because the firm, facing uncer-

[2]The main difference between the second and the third strands is that in situations of limited capacity firms mainly choose prices, while without capacity constraints firms choose their production quantities as well as prices.

[3]In Chen and Parlar (2005) an alternative model is considered, in which the probability that each consumer orders in advance is a beta-distributed random variable.

[4]An exception is Chu and Zhang (2011) in which the Newsvendor Problem is (implicitly) assumed away because

tain demand, has to choose its production quantity prior to the regular selling season. Obviously, learning from pre-orders benefits the firm because it helps to better forecast the demand in the regular selling season. Both issues are also central in our paper. Because we assume that the mean of the distribution of consumers' valuations is unknown to the retailer, learning in our model is not only on the consumer pool, but also on the distribution of consumers' valuations.

Like Zhao and Stecke (2010), Prasad, Stecke, and Zhao (2011), and Chu and Zhang (2011), consumers in our model are strategic. The key difference between our paper and existent literature is the introduction of experienced consumers into the model. As stated before, experienced consumers make the strategy of advance selling more attractive to the firm.

3 Model Setup

Consider a firm or a retailer who sells a product over two periods.[5] The first period is the advance selling season and the second period is the regular selling season. Any consumer who pre-orders in the first period is guaranteed delivery of the product in the second period. Those who do not pre-order can buy in the regular selling season, but there is a risk that the product will be out of stock. There are two types of consumers – experienced and inexperienced. Experienced consumers know their valuations (i.e., their willingness to pay for the product) from the outset, whereas inexperienced consumers learn their valuations only in the second period. Each consumer is willing to buy at most one unit of the product.

The number of experienced consumers is m_e. The number of inexperienced consumers, M_i, is a random variable; the distribution of M_i is lognormal $\mathrm{LN}\left(\nu_i, \tau_i^2\right)$ with the mean $m_i = \exp\left\{\nu_i + \tau_i^2/2\right\}$.[6] Both m_e and the distribution of M_i are common knowledge.

Consumers' valuations of the product are normally distributed with mean μ and variance σ^2 (i.e., $v \sim \mathrm{N}\left(\mu, \sigma^2\right)$). While all consumers know the distribution from which their valuations are drawn, the retailer does not. We model the retailer's uncertainty by assuming that μ is high, μ_H, with probability γ and low, μ_L, with probability $1-\gamma$.

The marginal production cost is c and the price during the regular selling season is p. For each unsold unit of the product at the end of the regular selling season the retailer gets its salvage value s. We assume $s < c < p$.

The retailer decides on advance selling price $x \leq p$ in the beginning of the advance selling season. After the conclusion of the advance selling season the retailer must decide how much to produce. Let D_1 denote the number of consumers who buy in the advance selling season. Then the retailer's quantity choice is $Q = D_1 + q$, where D_1 fulfills the pre-orders. Quantity q satisfies the (stochastic) demand during the regular selling season, denoted by D_2.

Table 2 lists the notation introduced above and also some of the notation introduced later. Figure 1 displays the timeline of the model. In the beginning of the first period, all consumers learn μ and all experienced consumers learn their valuations. During the first period, the retailer announces advance selling price x, then each consumer decides whether to pre-order. At the end of the first period, the retailer observes the number of pre-orders D_1, updates his forecast of

the salvage value equals the marginal cost.

[5]From now on we shall speak of the firm as the retailer.

[6]We use a lognormal distribution to avoid negative realizations of the number of inexperienced consumers.

Table 2: Notation

Parameters	
c	marginal cost
s	salvage value
p	price in the regular selling season
m_e	number of experienced consumers
Random variables	
$v \sim \mathrm{N}\left(\mu, \sigma^2\right)$	distribution of consumers' valuations
$\mu \in \{\mu_H, \mu_L\}$	two-point mass distribution, $\mathrm{Prob}(\mu_H) = \gamma$ and $\mathrm{Prob}(\mu_L) = 1 - \gamma$
$M_i \sim \mathrm{LN}\left(\nu_i, \tau_i^2\right)$	number of inexperienced consumers, mean $m_i = \exp\left\{\nu_i + \frac{\tau_i^2}{2}\right\}$
Decision variables	
q	quantity produced for the regular selling season
Q	total quantity produced (includes pre-orders)
x	advance selling price
Distribution and density functions	
$F(\cdot)$	cdf of $\mathrm{N}\left(\mu, \sigma^2\right)$, $F(y) = \Phi\left(\frac{y-\mu}{\sigma}\right)$
$\overline{F}(y)$	$= 1 - F(y)$
$f(\cdot)$	density function of $\mathrm{N}\left(\mu, \sigma^2\right)$, $f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}$
$G(\cdot)$	cdf of $\mathrm{LN}\left(\nu, \tau^2\right)$, $G(y) = \Phi\left(\frac{\ln y - \nu}{\tau}\right)$
$g(\cdot)$	density function of $\mathrm{LN}\left(\nu, \tau^2\right)$, $g(y) = \frac{1}{y\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(\ln y - \nu)^2}{2\tau^2}\right\}$
Other notation	
D_1, D_2	demands in the advance and regular selling seasons
π	retailer's expected profit from the regular selling season
Π	retailer's total expected profit (includes pre-orders)
η	stock-out probability
β	$= \frac{p-c}{p-s}$

the second-period demand D_2 and chooses production quantity Q. During the second period, all inexperienced consumers learn their valuations and those consumers who did not pre-order then decide whether to purchase the product at price p. The product is delivered at the end of the second period.

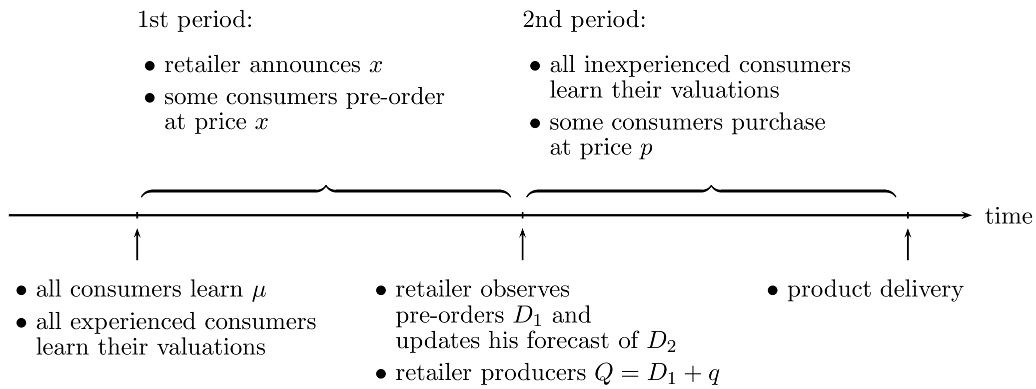


Figure 1: Timeline of the model

4 Equilibrium Analysis

Our goal in this section is to find the optimal advance selling price. To do this, we first derive consumers' optimal responses to any advance selling price, and how the retailer learns from the pre-orders and chooses his output. We then determine the endogenous stock-out probability and present the retailer's expected profit function.

4.1 Consumers' optimal purchasing decisions

Since experienced consumers know their valuations from the outset, they never wait until the regular selling season. Experienced consumers with valuations above x pre-order the product and pay discounted price $x \leq p$.

Inexperienced consumers do not know their valuations in the advance selling season. An inexperienced consumer has two options. The first is to pre-order and pay x. In this case the consumer's expected payoff is

$$\mu - x.$$

The other option is to wait until the regular selling season. The consumer learns her valuation v and purchases the product (provided it is in stock) if $v \geq p$. Her expected payoff is

$$(1-\eta)\int_p^{+\infty}(v-p)f(v)\,\mathrm{d}v,$$

where η is the stock-out probability and $f(\cdot)$ is the density function of $\mathrm{N}\left(\mu,\sigma^2\right)$, from which the consumer's valuation is drawn. The stock-out probability is the probability that the consumer will not be able to get the product when she actually wants to purchase it.

Thus, inexperienced consumers pre-order if and only if

$$\mu - x \geq (1-\eta)\int_p^{+\infty}(v-p)f(v)\,\mathrm{d}v,$$

or, equivalently,

$$x \leq \mu - (1-\eta) \int_{p}^{+\infty} (v-p) f(v) \, \mathrm{d}v.$$

Let

$$x_L \equiv \mu_L - (1-\eta) \int_{p}^{+\infty} (v-p) f_L(v) \, \mathrm{d}v \tag{1}$$

and

$$x_H \equiv \mu_H - (1-\eta) \int_{p}^{+\infty} (v-p) f_H(v) \, \mathrm{d}v \tag{2}$$

denote the threshold values for $\mu = \mu_L$ and $\mu = \mu_H$, respectively. Here $f_L(\cdot)$ is the density function of $\mathrm{N}\left(\mu_L, \sigma^2\right)$ and $f_H(\cdot)$ is the density function of $\mathrm{N}\left(\mu_H, \sigma^2\right)$. In Subsection 4.4 we show that the endogenously determined η is the same in (1) and (2). The explicit expressions for x_L and x_H (derived in Appendix) are

$$x_L = \mu_L - (1-\eta) \left((\mu_L - p) \overline{F}_L(p) + \sigma^2 f_L(p) \right)$$

and

$$x_H = \mu_H - (1-\eta) \left((\mu_H - p) \overline{F}_H(p) + \sigma^2 f_H(p) \right),$$

where $F_L(\cdot)$ and $F_H(\cdot)$ are the cumulative distribution functions of $\mathrm{N}\left(\mu_L, \sigma^2\right)$ and $\mathrm{N}\left(\mu_H, \sigma^2\right)$, respectively, and $\overline{F}(\cdot) = 1 - F(\cdot)$.

Lemma 1 (Properties of x_L and x_H)**.** *The threshold values $x_L(\eta, \sigma)$ and $x_H(\eta, \sigma)$ possess the following properties:*

(i) $x_L(\eta, \sigma) < x_H(\eta, \sigma)$ *for all η and σ;*

(ii) $\partial x_L / \partial \sigma < 0$ *and* $\partial x_H / \partial \sigma < 0$*;*

(iii) $\partial x_L / \partial \eta > 0$ *and* $\partial x_H / \partial \eta > 0$*.*

Since $x_L < x_H$ always holds, we consider the following three regions for advance selling price x.

- Region A: $x \leq x_L$. All inexperienced consumers pre-order.
- Region B: $x_L < x \leq x_H$. Inexperienced consumers pre-order if $\mu = \mu_H$.
- Region C: $x > x_H$. All inexperienced consumers wait until the second period.

Properties (ii) and (iii) will be useful for our sensitivity analysis in Section 5.

4.2 Learning by the retailer from pre-orders

When advance selling price x is in region A, experienced consumers with valuations above x and all inexperienced consumers pre-order. No one will wait until the regular selling season. As the retailer's forecast of the second-period demand is $D_2 = 0$, he produces $Q = D_1$.

When x is in region B, the retailer learns μ through observing D_1. If $D_1 = m_e\overline{F}_L(x)$, then the retailer infers that $\mu = \mu_L$. The retailer produces $Q = D_1 + q$, where q satisfies the second-period demand that comprises of inexperienced consumers with valuations above p,

$$D_2 = M_i \operatorname{Prob}(v > p) = M_i\overline{F}_L(p).$$

Because $M_i \sim \mathrm{LN}\left(\nu_i, \tau_i^2\right)$, it is straightforward to show that

$$D_2 \sim \mathrm{LN}\left(\nu_i + \ln \overline{F}_L(p), \tau_i^2\right).$$

If $D_1 \neq m_e\overline{F}_L(x)$, the retailer infers $\mu = \mu_H$ and $D_2 = 0$, hence produces $Q = D_1$. (Like in region A, no one waits until the regular selling season – experienced consumers with valuations above x and all inexperienced consumers pre-order.)

In region C the retailer also learns μ. If $D_1 = m_e\overline{F}_L(x)$, then the retailer infers that $\mu = \mu_L$. The retailer produces $Q = D_1 + q$, where q is for the second-period demand

$$D_2 = M_i\overline{F}_L(p) \sim \mathrm{LN}\left(\nu_i + \ln \overline{F}_L(p), \tau_i^2\right).$$

If $D_1 = m_e\overline{F}_H(x)$, the retailer knows $\mu = \mu_H$. The retailer produces $Q = D_1 + q$, where q is for the second-period demand

$$D_2 = M_i\overline{F}_H(p) \sim \mathrm{LN}\left(\nu_i + \ln \overline{F}_H(p), \tau_i^2\right).$$

4.3 Optimal value of q

In regard to the retailer's choice of q, it remains to find the optimal q when D_2 follows the distribution $\mathrm{LN}\left(\nu_i + \ln \overline{F}_L(p), \tau_i^2\right)$ and when it follows $\mathrm{LN}\left(\nu_i + \ln \overline{F}_H(p), \tau_i^2\right)$.

For any D_2, if q units are produced, then $\min\{q, D_2\}$ units are sold and $(q - D_2)^+ = \max\{q - D_2, 0\}$ are salvaged. The retailer's expected profit from the second period, denoted by π, is

$$\pi(q) = p\,\mathrm{E}\left[\min\{q, D_2\}\right] + s\,\mathrm{E}\left[(q - D_2)^+\right] - cq. \tag{3}$$

The problem of maximizing (3) is the Newsvendor Problem, well-known in the operations management literature. Using the fact that $\min\{q, D_2\} = D_2 - (D_2 - q)^+$, we can rewrite the retailer's expected profit as

$$\pi(q) = \mathrm{E}[D_2](p - c) - \mathrm{E}\left[(D_2 - q)^+\right](p - c) - \mathrm{E}\left[(q - D_2)^+\right](c - s).$$

The optimal value of q, therefore, minimizes the expected underage and overage cost

$$\mathrm{E}\left[(D_2 - q)^+\right](p - c) + \mathrm{E}\left[(q - D_2)^+\right](c - s).$$

The first-order condition is

$$\operatorname{Prob}(D_2 \leq q^*) = \beta,$$

where

$$\beta \equiv \frac{p-c}{p-s}.$$

It is clear that q^* selected this way increases in β and therefore increases in the per unit underage cost $p-c$ and decreases in the per unit overage cost $c-s$.

For the lognormal distribution $D_2 \sim \mathrm{LN}\left(\nu, \tau^2\right)$ the optimal production quantity is given by

$$q^* = \exp\{\nu + \tau z_\beta\} \tag{4}$$

and

$$\pi(q^*) = (p-s)\left(1 - \Phi(\tau - z_\beta)\right) \exp\left\{\nu + \frac{\tau^2}{2}\right\}, \tag{5}$$

where z_β is the β-th percentile of the standard normal distribution, $z_\beta \equiv \Phi^{-1}(\beta)$. See Appendix for derivations of (4) and (5).

Applying (4) and (5) to $D_2 \sim \mathrm{LN}\left(\nu_i + \ln \overline{F}_L(p), \tau_i^2\right)$, the optimal q, denoted by q_L^*, and the resulting expected profit π_L are

$$q_L^* = \exp\{\nu_i + \tau_i z_\beta\} \overline{F}_L(p)$$

and

$$\pi_L = (p-s)\left(1 - \Phi(\tau_i - z_\beta)\right) m_i \overline{F}_L(p).$$

Similarly, under $D_2 \sim \mathrm{LN}\left(\nu_i + \ln \overline{F}_H(p), \tau_i^2\right)$ the optimal q, denoted by q_H^*, and the resulting expected profit π_H are

$$q_H^* = \exp\{\nu_i + \tau_i z_\beta\} \overline{F}_H(p)$$

and

$$\pi_H = (p-s)\left(1 - \Phi(\tau_i - z_\beta)\right) m_i \overline{F}_H(p).$$

4.4 Stock-out probability

Given D_2, the (conditional) probability of any consumer who wants to purchase the product in the regular selling season but is unable to get it is the fraction of excess demand,

$$\mathrm{Prob}(\text{stock-out}|D_2) = \left(\frac{D_2 - q^*}{D_2}\right)^+ = \begin{cases} 0, & D_2 \le q^*, \\ \frac{D_2 - q^*}{D_2}, & D_2 > q^*. \end{cases}$$

Hence, the stock-out probability is the expected value of this expression over the distribution of D_2,

$$\eta = \mathrm{E}\left[\left(\frac{D_2 - q^*}{D_2}\right)^+\right]. \tag{6}$$

For $D_2 \sim \mathrm{LN}\left(\nu, \tau^2\right)$,

$$\eta = \int_{q^*}^{+\infty} \frac{D_2 - q^*}{D_2} g(D_2)\, \mathrm{d}D_2,$$

where $g(\cdot)$ is the density function of $\text{LN}\left(\nu, \tau^2\right)$ and q^* is given in (4). The explicit expression for the stock-out probability is obtained in Appendix,

$$\eta = 1 - \beta - \exp\left\{\tau z_\beta + \frac{\tau^2}{2}\right\}\left(1 - \Phi(z_\beta + \tau)\right). \tag{7}$$

Note that the expression in (7) is independent of ν. Accordingly, the same η results from $D_2 \sim \text{LN}\left(\nu_i + \ln \overline{F}_L(p), \tau_i^2\right)$ and $D_2 \sim \text{LN}\left(\nu_i + \ln \overline{F}_H(p), \tau_i^2\right)$. This finding is presented in Lemma 2.

Lemma 2 (Stock-out probability η)**.** *The stock-out probability under the second-period demand $D_2 \sim \text{LN}\left(\nu_i + \ln \overline{F}_L(p), \tau_i^2\right)$ is equal to that under $D_2 \sim \text{LN}\left(\nu_i + \ln \overline{F}_H(p), \tau_i^2\right)$ and is given by*

$$\eta = 1 - \beta - \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\}\left(1 - \Phi(z_\beta + \tau_i)\right).$$

It is important to note that the stock-out probability in our model is endogenously determined, because q^* is the optimal choice. It follows that $\eta < 1 - \beta$, which is expected, as

$$\begin{aligned}\eta = \text{E}\left[\left(\frac{D_2 - q^*}{D_2}\right)^+\right] &< [\mathbb{1}(D_2 > q^*)] = \text{Prob}(D_2 > q^*) \\ &= 1 - \text{Prob}(D_2 < q^*) = 1 - \beta,\end{aligned}$$

where $\mathbb{1}(D_2 > q^*)$ is the indicator of the corresponding event, and

$$\left(\frac{D_2 - q^*}{D_2}\right)^+ < \mathbb{1}(D_2 > q^*)$$

for all D_2.[7]

Lemma 3 (Properties of η)**.** *The stock-out probability $\eta = \eta(\beta, \tau_i)$ possesses the following properties:*

(i) $\partial\eta/\partial\tau_i > 0$*,* $\eta(\beta, 0) = 0$*, and* $\lim_{\tau_i \to +\infty} \eta(\beta, \tau_i) = 1 - \beta$*;*

(ii) $\partial\eta/\partial\beta < 0$*,* $\eta(0, \tau_i) = 1$*, and* $\eta(1, \tau_i) = 0$*.*

Combining the results of Lemma 1(iii) and Lemma 3 we can conclude that the threshold values x_L and x_H increase in τ_i and decrease in β.

[7] Another theoretical study that models explicitly the risk of stock out that consumers face is Prasad, Stecke, and Zhao (2011). In their study the "stocking out probability" (defined on page 5) has the same meaning as η in our model: a consumer who waited until the regular selling season will not be able to get the product with probability η. However, instead (6), they use the expression $\eta = \text{Prob}(D_2 > q^*)$, which yields $\eta = 1 - \beta$ (page 7). Their formula, however, does not account for the fact that when $D_2 > q^*$, the consumer might still get the product.

4.5 The retailer's expected profit

We can now write the retailer's expected total profit Π as a function of advance selling price x. The part of the retailer's expected profit that comes from experienced consumers equals

$$m_e \left(\gamma \overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x-c),$$

as experienced consumers never wait until the regular selling season and only those with valuations above x (fraction $\overline{F}_H(x)$ in the case $\mu = \mu_H$ and fraction $\overline{F}_L(x)$ in the case $\mu = \mu_L$) purchase the product in the advance selling season.

The purchasing behavior of inexperienced consumers depends on the region that x belongs to. If $x \le x_L$ (region A), then all inexperienced consumers pre-order. Hence,

$$\Pi^A(x) = m_e \left(\gamma \overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x-c) + m_i(x-c). \tag{8}$$

Next, consider $x_L < x \le x_H$ (region B). In the case $\mu = \mu_H$ all inexperienced pre-order, yielding $m_i(x-c)$ to the retailer. In the case $\mu = \mu_L$ all inexperienced consumers wait until the second period, yielding π_L (calculated in Section 4.3) to the retailer. Hence,

$$\Pi^B(x) = m_e \left(\gamma \overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x-c) + \gamma m_i(x-c) + (1-\gamma)\pi_L. \tag{9}$$

Finally, if $x > x_H$ (region C), then all inexperienced consumers wait until the second period. Since the retailer learns μ in the first period, his expected payoff from inexperienced consumers is $\gamma\pi_H + (1-\gamma)\pi_L$. Hence,

$$\Pi^C(x) = m_e \left(\gamma \overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x-c) + \gamma\pi_H + (1-\gamma)\pi_L. \tag{10}$$

The retailer's total expected profit as a function of advance selling price x is, therefore,

$$\Pi(x) = \begin{cases} \Pi^A(x), & x \le x_L, \\ \Pi^B(x), & x_L < x \le x_H, \\ \Pi^C(x), & x > x_H. \end{cases}$$

4.6 Advance selling vs. no advance selling

Before moving onto the optimal advance selling price for the retailer, we explore next whether there always is an incentive for the retailer to implement the advance selling strategy.

Without advance selling, the retailer sells in the regular selling season at price p. Let $\Pi_H(Q)$ and $\Pi_L(Q)$ denote the expected profits as functions of production quantity Q for $\mu = \mu_H$ and $\mu = \mu_L$, respectively. Let Q_0 denote the optimal quantity. Then, without advance selling, the retailer's maximum expected profit is

$$\Pi^0 = \gamma\Pi_H(Q_0) + (1-\gamma)\Pi_L(Q_0).$$

We will show that advance selling at price p leads to a higher expected profit than no advance selling, that is, $\Pi(p) > \Pi^0$. It follows that the retailer's expected profit under advance selling (at the optimal price which may or may not be equal to p) must be greater than that under no advance selling.

Advance selling brings two benefits for the retailer: learning the true value of μ before choosing production quantity and receiving precommitted orders. We show next that the first benefit alone improves the retailer's expected profit over that under no advance selling. Let Q_H and Q_L denote the optimal quantities for $\mu = \mu_H$ and $\mu = \mu_L$, respectively. It follows that $\Pi_H(Q_H) \geq \Pi_H(Q_0)$ and $\Pi_L(Q_L) \geq \Pi_L(Q_0)$ with at least one in strict inequality. Accordingly,

$$\gamma\Pi_H(Q_H) + (1-\gamma)\Pi_L(Q_L) > \Pi^0.$$

This inequality indicates that knowing the true value of μ before choosing Q is superior to choosing Q without knowing the true value of μ. Since $\Pi(p)$ incorporates the benefit from possible pre-orders, we must have

$$\Pi(p) \geq \gamma\Pi_H(Q_H) + (1-\gamma)\Pi_L(Q_L).$$

Hence, $\Pi(p) > \Pi^0$.

Therefore, we have the following proposition.[8]

Proposition 1 (Advance selling vs. no advance selling)**.** *Advance selling is always superior to no advance selling for the retailer.*

4.7 Optimal advance selling price

For the rest of our analysis, we will assume that $c < x_L < x_H < p$ holds.[9] Furthermore, we make the following simplifying assumption.

Assumption 1. *The function*

$$\left(\gamma\overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x-c)$$

increases in x on $[c,p]$.

This assumption implies that the expected profit from experienced consumers,

$$m_e\left(\gamma\overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x-c),$$

is an increasing function of x for all $x \in [c,p]$. It follows that, as far as experienced consumers are concerned, the retailer has no incentives to offer an advance selling discount. Accordingly, if discounting for pre-orders is offered it must due to the presence of inexperienced consumers.

It is easy to see that under Assumption 1 the retailer's expected profit $\Pi(x)$ increases in x in each of the three regions A, B, and C. This does not imply, however, that $\Pi(x)$ increases in x

[8] We do not consider the cost of adopting advance selling in this paper. Proposition 1 holds as long as the adoption cost is lower than $\Pi(p) - \Pi^0$.

[9] There are six possible relationships between c, p, x_L, and x_H: $c < p < x_L < x_H$, $c < x_L < p < x_H$, $c < x_L < x_H < p$, $x_L < c < p < x_H$, $x_L < c < x_H < p$, and $x_L < x_H < c < p$.

on $[c, p]$, as $\Pi(x)$ can jump down at $x = x_L$ and/or at $x = x_H$. A jump down at x_L occurs if and only if $\Pi^A(x_L) > \Pi^B(x_L)$.[10] That is,

$$(1-\gamma)m_i(x_L - c) > (1-\gamma)\pi_L,$$

or equivalently,

$$x_L - c > (p-s)\,(1 - \Phi(\tau_i - z_\beta))\,\overline{F}_L(p).$$

Similarly, a jump down at x_H occurs if and only if $\Pi^B(x_H) > \Pi^C(x_H)$.[11] That is,

$$\gamma m_i(x_H - c) > \gamma \pi_H,$$

or equivalently,

$$x_H - c > (p-s)\,(1 - \Phi(\tau_i - z_\beta))\,\overline{F}_H(p).$$

Hence, we have the following four patterns for $\Pi(x)$ (see Figure 2).

- Pattern 1: $\Pi(x)$ jumps down at both x_L and x_H.
- Pattern 2: $\Pi(x)$ jumps up at both x_L and x_H.
- Pattern 3: $\Pi(x)$ jumps up at x_L, jumps down at x_H.
- Pattern 4: $\Pi(x)$ jumps down at x_L, jumps up at x_H.

Regarding the optimal advance selling price x^*, under pattern 1 it is either x_L, x_H, or p; under pattern 2 it is p; under pattern 3, it is either x_H or p; under pattern 4, it is either x_L or p. Our extensive numerical simulations indicate that all four patterns may arise and under each pattern the optimal advance selling price can take any of the possible values mentioned above. It follows that the retailer's optimal advance selling price can be any one of the three values: x_L, x_H, or p.

Proposition 2 (Optimal advance selling price)**.** *Under Assumption 1, the optimal advance selling price x^* is either x_L, x_H, or p.*

The three likely optimal advance selling prices reflect two different tradeoffs for the retailer: between low price-high sales and high price-low sales, and between low price-low expected overage and underage cost and high price-high expected overage and underage cost. Under all three prices, experienced consumers only buy in the advance selling season. Hence, the tradeoff for the retailer from this group of consumers is only between a lower price and therefore higher expected sales and a higher price and lower expected sales.

For the inexperienced group of consumers, let us first focus on the comparison between x_L and p. At x_L all inexperienced consumers pre-order while at p none pre-order and only those with realized values above p will buy in the regular selling season. Hence, both tradeoffs described above are present here. First, x_L corresponds to higher sales and a lower price, while p corresponds to much lower sales and a higher price. Second, x_L means zero overage and underage cost, while p leads to a positive expected overage and underage cost.

[10] Since $\Pi_B(\cdot)$ is defined on $(x_L, x_H]$, $\Pi^B(x_L)$ is the limiting value.
[11] Since $\Pi_C(\cdot)$ is defined on $(x_H, p]$, $\Pi^C(x_H)$ is the limiting value.

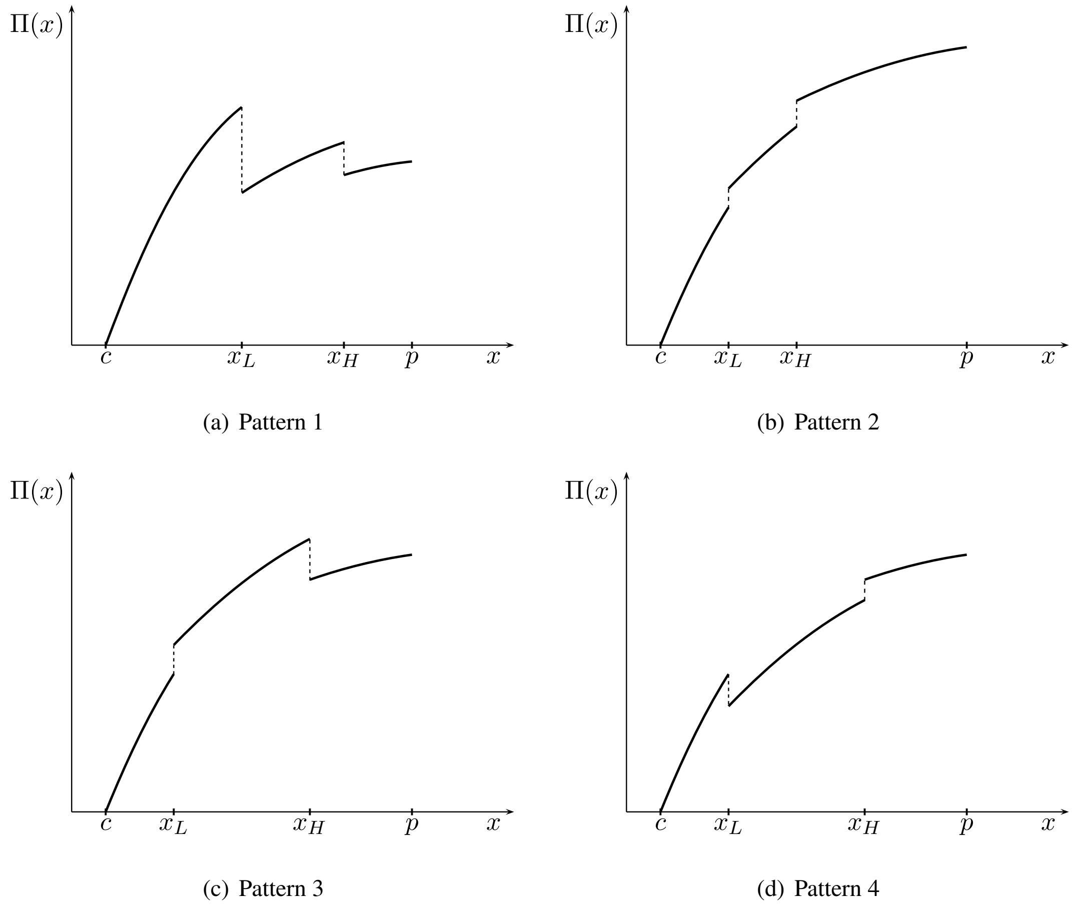


Figure 2: The retailer's expected profit as a function of x

Adding the intermediate price x_H to the mix, the same two tradeoffs are involved between each pair of these prices. For example, moving from x_L to x_H, the expected sales decrease while the expected overage and underage cost increases, both are due to the fact that there is a positive probability that at x_H all inexperienced consumers will wait until the regular selling season. Comparing all three possible advance selling prices, we conclude that, as the advance selling price changes from x_L to x_H to p, the expected sales decrease and the expected overage and underage cost increases.

5 Sensitivity Analysis

In this section we consider how the retailer's optimal advance selling price x^* and the expected profit $\Pi(x^*)$ are affected by some important parameters of the model. Let $\Pi^* \equiv \Pi(x^*)$. Intuitively, we should expect that a decrease in the salvage value s or an increase in demand uncertainty τ_i would result in lower Π^*. Surprisingly, we find that Π^* might actually increase

(Subsection 5.1).

We are also interested in how consumer characteristics – the relative number of experienced consumers (measured by parameter α introduced below) and valuation uncertainty of inexperienced consumers (captured by σ) – affect the retailer. Will a decrease in α and/or an increase in σ lead to lower Π^*? As we show in Subsections 5.2 and 5.3, both lower and higher Π^* are possible.

In all of our sensitivity analysis in this section, we focus on small changes in the parameter values so that the retailer's optimal choice stays the same in that it does not jump to one of the other two points. Although we work with τ_i and σ, the same results apply to small changes in τ_i^2 and σ^2.

In Table 3 all of the directional changes in the parameters (the first row) are chosen to "hurt" the retailer on an intuitive basis. Therefore, all cases in which Π^* increases ($\Pi^* \uparrow$) represent counterintuitive results.

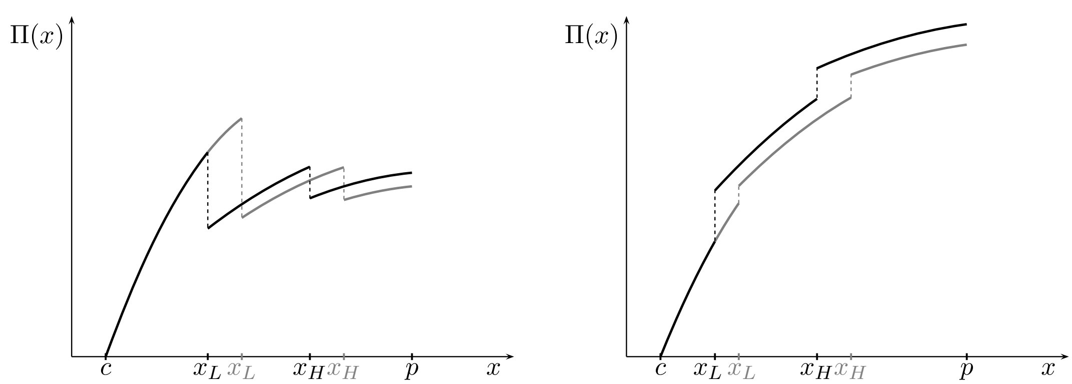
5.1 Sensitivity analysis – s and τ_i

In this subsection we consider how a decrease in the salvage value s and an increase in demand uncertainty τ_i affects the retailer.

We first focus on s. Suppose s decreases. Then $\beta = (p-c)/(p-s)$ decreases. By Lemma 3(ii), η increases. An increase in η positively affects x_L and x_H. As to π_L and π_H, they go down as s decreases. Indeed, applying the Envelope Theorem to (3) yields

$$\left.\frac{\partial \pi}{\partial s}\right|_{q=q^*} = \mathrm{E}\left[(q^* - D_2)^+\right] > 0.$$

Geometrically, a decrease in s shifts the thresholds x_L and x_H to the right and the curves $\Pi^B(x)$ and $\Pi^C(x)$ defined in (9) and (10) down.



(a) Pattern 1

(b) Pattern 2

Figure 3: The effects of a decrease in s (increase in τ_i) on $\Pi(x)$

Consider, for example, pattern 1 and suppose $x^* = x_L$. In Figure 3(a) the black and gray curves represent, respectively, $\Pi(x)$ before and after a decrease in s. It is easy to see that both x^* and Π^* increase. If $x^* = x_H$, then a decrease in s leads to higher x^*; Π^* can increase or decrease. If $x^* = p$, then a decrease in s leads to lower Π^*.

Consider next pattern 2. The optimal advance selling price is p. As can be seen from Figure 3(b), a decrease in s leads to a lower Π^*. Similar reasoning applies to patterns 3 and 4. It is important to point out that the same results on the directional changes of x^* and Π^* hold across all four patterns as long as the same optimal choice is obtained. The second column of Table 3 reports the sensitivity results on x^* and Π^* in terms of s.

Table 3: Sensitivity analysis results

optimal choice x^* \ parameter change	$s\downarrow$	$\tau_i \uparrow$	$\alpha\downarrow$	$\sigma\uparrow$
x_L	$x^*\uparrow$ $\Pi^*\uparrow$	$x^*\uparrow$ $\Pi^*\uparrow$	x^* unchanged $\Pi^*\uparrow$	$x^*\downarrow$ $\Pi^*\downarrow$
x_H	$x^*\uparrow$ $\Pi^*\uparrow\downarrow$	$x^*\uparrow$ $\Pi^*\uparrow\downarrow$	x^* unchanged $\Pi^*\uparrow\downarrow$	$x^*\downarrow$ $\Pi^*\uparrow\downarrow$
p	x^* unchanged $\Pi^*\downarrow$	x^* unchanged $\Pi^*\downarrow$	x^* unchanged $\Pi^*\downarrow$	x^* unchanged $\Pi^*\uparrow\downarrow$

What is the effect of an increase in τ_i on x^* and Π^*? By Lemma 3(i), η increases. As a result, x_L and x_H increase. Clearly, an increase in τ_i negatively affects π_L and π_H. It follows that an increase in τ_i affects x^* and Π^* in similar ways as a decrease in s. (See the third column of Table 3.)

The intuition for the above results in regard to a change in the salvage value s is as follows (similar for an increase in τ_i). When s becomes lower, the per unit overage cost becomes higher and therefore the retailer reduces his output to avoid too much unsold product at the end of the regular selling season. This raises the stock-out probability η for consumers who wait until the regular selling season. As a result, inexperienced consumers are willing to pay a higher advance selling price (i.e., higher x_L and x_H) to secure delivery of the product. Consider the case in which the retailer's optimal advance selling price $x^* = x_L$. Since the retailer's expected profit from the group of experienced consumers is an increasing function of x (Assumption 1), it becomes higher. Obviously, the retailer's expected profit from the group of inexperienced consumers rises as well since all of them pre-order at a higher price. Hence, the total expected profit of the retailer becomes greater. In this case, we obtain the counterintuitive result that the retailer can benefit from a decrease in s.

The increased stock-out probability does bite the retailer if the optimal advance selling price $x^* = p$. In this case, the retailer's expected profit from experienced consumers remains the same as there is no change for this group both in price and in the number of buyers. However, the retailer's expected profit from the group of inexperienced consumers, who all wait until the regular selling season, becomes lower due to the increased per unit overage cost. It follows that the retailer is hurt by a decrease in s.

Finally, consider $x^* = x_H$. With probability γ the retailer will benefit from a decrease in s (similar to the case $x^* = x_L$) and with probability $1-\gamma$ the retailer will be hurt by a decrease in s (similar to the case $x^* = p$). As a result, depending on these probabilities and the respective gain and loss, both directions of change are possible for the retailer's total expected profit.

5.2 Sensitivity analysis – α

Let $m = m_e + m_i$ denote the total expected number of (experienced and inexperienced) consumers, and let α denote the proportion of experienced consumers in the market. Thus, $m_e = \alpha m$ and $m_i = (1-\alpha)m$. In this subsection we consider how a decrease in α affects the retailer.

Note that η, x_L and x_H are independent of α. Hence, x^* stays the same. To see how the retailer's expected profit is affected, we substitute $m_e = \alpha m$ and $m_i = (1-\alpha)m$ into the expressions (8) through (10), remembering that π_L and π_H depend on m_i. In Appendix we calculate the derivatives of $\Pi^A(x_L)$, $\Pi^B(x_H)$, and $\Pi^C(p)$ with respect to α, and show that the first derivative is negative, the second can be positive or negative, and the third is positive. Therefore, if $x^* = x_L$ then a decrease in α leads to higher Π^*, and if $x^* = p$ – to lower Π^*. If $x^* = x_H$, Π^* can increase or decrease.

The fourth column of Table 3 reports the results of our sensitivity analysis with respect to α. The intuition is straightforward. A decrease in α changes the composition of experienced and inexperienced consumers in the total consumer population by decreasing the group size of experienced consumers and increasing the group size of inexperienced consumers. Such a change does not alter the incentive for the retailer to produce to satisfy the demand in the regular selling season and does not affect each individual consumer's incentive to pre-order in the advance selling season. That is, η, x_L and x_H all remain unchanged. It does, however, affect the retailer's expected profit. In the case in which the retailer's optimal advance selling price $x^* = x_L$, no one buys in the regular selling season. The size of pre-orders placed by experienced consumers decreases while the size of pre-orders placed by inexperienced consumers increases. The latter change is greater than the former, because only a fraction of experienced consumers purchase the product. Hence, the retailer's expected profit increases.

If the optimal advance selling price $x^* = p$, the opposite occurs. In this case all experienced consumers whose valuations are above p pre-order but only a fraction of those inexperienced consumers whose valuations are above p get to buy the product in the regular selling season due to the positive stock-out probability. This leads to a net loss in the expected sales for the retailer and therefore to a lower expected profit.

In the case in which the optimal advance selling price $x^* = x_H$, the retailer's expected profit rises with probability γ (corresponding to all inexperienced consumers pre-ordering like in the case $x^* = x_L$) and falls with probability $1-\gamma$ (corresponding to all inexperienced consumers waiting to buy in the regular selling season like in the case $x^* = p$). As a result, depending on these probabilities and the respective gain and loss, both directions of change are possible for the retailer's total expected profit.

5.3 Sensitivity analysis – σ

Parameter σ captures variation in consumer valuations. For inexperienced consumers an increase in σ also means they become more uncertain about their valuations.

By Lemma 1(ii), both x_L and x_H decrease as σ increases. In Appendix we show that whenever $x^* = x_L$, an increase in σ leads to lower expected profit for the retailer. If $x^* = x_H$ or p, Π^* can increase or decrease (we constructed numerical examples that show that both directions are possible). The fifth column of Table 3 reports the results.

The intuition is as follows. When σ increases, inexperienced consumers become less certain about their valuations of the product. As a result, they require better incentives than before to be willing to pre-order in the advance selling season (i.e., lower x_L and x_H). In the case in which the optimal advance selling price $x^* = x_L$, all consumers who buy pre-order, now at a lower price. By Assumption 1, the retailer's expected profit from experienced consumers becomes lower. The retailer's expected profit from inexperienced consumers falls as well since all of them pre-order at a lower price. Hence, the total expected profit of the retailer becomes smaller.

Consider now the case in which the optimal advance selling price $x^* = p$. As σ increases the valuation distribution functions become more dispersed in that more consumers have valuations farther away from the mean value than before. As a result, if $p > \mu$ then more consumers have valuations above p and if $p < \mu$ then less consumers have valuations above p. Suppose $p > \mu_H$, which also implies $p > \mu_L$. The retailer gets more pre-orders from experienced consumers in the advance selling season and a larger demand from inexperienced consumers in the regular selling season, both yielding greater profit. On the other hand, for $p < \mu_L$ (hence, $p < \mu_H$), the opposite occurs and the retailer's total expected profit decreases. Obviously, either an increase or a decrease in the retailer's expected profit is possible if $\mu_L < p < \mu_H$. Thus, in the case $x^* = p$ the directional change in the retailer's expected profit depends on the value of p in relation to μ_L and μ_H, all of which are exogenously given in our model.

Finally, consider the case in which the optimal advance selling price $x^* = x_H$. The retailer's total expected profit falls with probability γ (corresponding to all inexperienced consumers pre-ordering like in the case $x^* = x_L$) and falls or rises with probability $1 - \gamma$ (corresponding to all inexperienced consumers waiting to buy in the regular selling season like in the case $x^* = p$). As a result, depending on these probabilities (γ and $1-\gamma$) and the respective gain and loss, both directions of change are possible for the retailer's total expected profit.

6 Extensions

In this section, the equilibrium analysis in Section 4 is extended in two directions. We first relax the assumption that $c < x_L < x_H < p$ so as to explore the possibility that the retailer sets an advance selling price that is below cost. Then, we discuss the retailer's optimal pricing strategy without Assumption 1.

6.1 Advance selling below cost ($x < c$)

Based on our analysis in Section 4, only if the assumption that $c < x_L < x_H < p$ is relaxed can it become possible that the optimal advance selling price x is less than c. Accordingly, we examine the possibility of $x^* < c$ under the following two scenarios: $x_L < c < x_H < p$ and $x_L < x_H < c < p$. Following the same arguments as those presented in Section 4, we have that the optimal advance selling price in these two scenarios must take one of the three values: x_L, x_H, and p.

Can x_L be the optimal advance selling price? Since

$$\Pi^A(x_L) = m_e \left(\gamma \overline{F}_H(x_L) + (1-\gamma)\overline{F}_L(x_L)\right)(x_L - c) + m_i(x_L - c) < 0,$$

setting $x = x_L$ would lead to a negative expected profit under either scenario. It is therefore inferior to setting the advance selling price at p, which implies a positive expected profit. Thus, the optimal advance selling price can never be x_L in either of the above two scenarios.

In the scenario $x_L < c < x_H < p$, setting $x = x_H$ does not imply pricing under cost. So the question becomes, can x_H be the optimal advance selling price in the scenario $x_L < x_H < c < p$? We have

$$\begin{aligned}\Pi^B(x_H) &= m_e \left(\gamma \overline{F}_H(x_H) + (1-\gamma)\overline{F}_L(x)\right)(x_H - c) + \gamma m_i(x_H - c) + (1-\gamma)\pi_L \\ &< m_e \left(\gamma \overline{F}_H(p) + (1-\gamma)\overline{F}_L(p)\right)(p - c) + \gamma \pi_H + (1-\gamma)\pi_L = \Pi^C(p).\end{aligned}$$

Thus, the optimal advance selling price cannot be x_H.

In conclusion, we have shown above that the retailer's optimal advance selling price is never below the unit production cost c.

6.2 Interior optimum

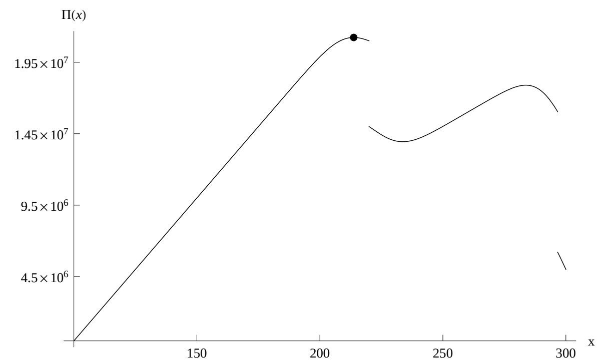
Our study of the retailer's optimal advance selling price and the subsequent sensitivity analysis were based on the simplifying Assumption 1 that $\left(\gamma \overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x - c)$ increases in x on $[c, p]$. In this subsection, we wish to point out that our main results continue to hold if Assumption 1 is not maintained. In particular, the following numerical examples demonstrate that the retailer's optimal advance selling price can occur in all three relevant regions, although it is in general an interior optimum in the respective region (Figure 4).

In these examples, we present three different interior optimal choices by the retailer that are located in the three respective regions. In all three cases, $c = 100$, $m = 200000$, $\alpha = 0.5$, and $\gamma = 0.5$.

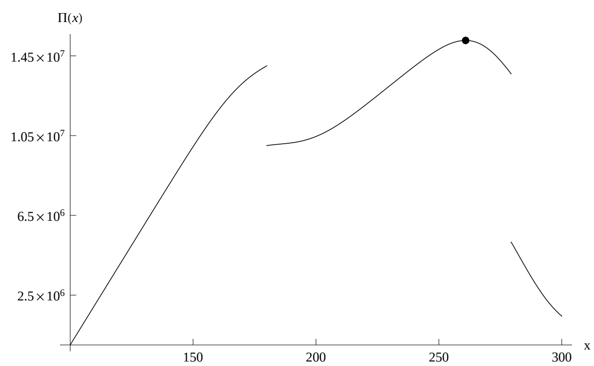
Example 1. *We use* $p = 300$, $s = 45$, $\sigma = 10$, $\mu_L = 220$, $\mu_H = 300$, *and* $\tau_i = 4.5$. *In this example, the endogenous values are* $\eta = 0.16$, $x_L = 220$, *and* $x_H = 296.66$*; the optimal advance selling price* $x^* = 213.75$ *is located in region A. This example is illustrated in Figure 4(a).*

Example 2. *We use* $p = 300$, $s = 60$, $\sigma = 15$, $\mu_L = 180$, $\mu_H = 280$, *and* $\tau_i = 1$. *In this example, the endogenous values are* $\eta = 0.06$, $x_L = 180$, *and* $x_H = 279.40$*; the optimal advance selling price* $x^* = 260.88$ *is located in region B. This example is illustrated in Figure 4(b).*

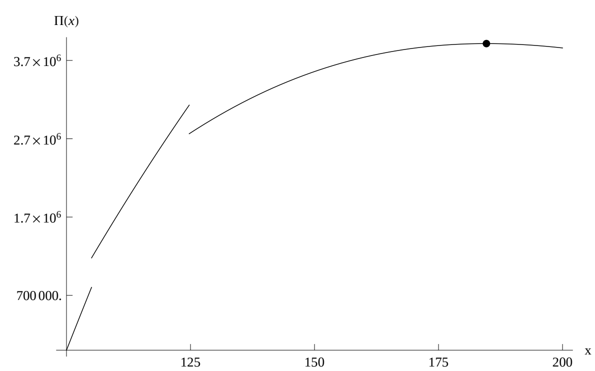
Example 3. *We use* $p = 200$, $s = 55$, $\sigma = 100$, $\mu_L = 115$, $\mu_H = 140$, *and* $\tau_i = 0.65$. *In this example, the endogenous values are* $\eta = 0.10$, $x_L = 105.05$, *and* $x_H = 124.74$*; the optimal advance selling price* $x^* = 184.65$ *is located in region C. This example is illustrated in Figure 4(c).*



(a) $c < x^* < x_L$



(b) $x_L < x^* < x_H$



(c) $x_H < x^* < p$

Figure 4: Interior optimum

7 Concluding Remarks

This paper has studied advance selling when the firm faces inexperienced consumers as well as a group of experienced consumers who have prior experience with an earlier version of the product. We find that it is always in the firm's best interest to adopt advance selling. The optimal pre-order price may or may not be at a discount to the regular selling price.

A number of issues are worthy of further investigation. Our model has no bearing on the adoption of advance selling by a firm selling a brand new product. An interesting follow-up study is to investigate what happens if the set of experienced consumers is empty. We anticipate that advance selling may still be optimal for the firm in some circumstances. Obviously, an advance selling discount becomes more likely in order to provide inexperienced consumers with an incentive to pre-order.

Another issue is the possibility of a price premium for pre-orders, which has not been analyzed in the present paper. As experienced consumers know their valuations of the product, some of them may be willing to pay a price premium so as to avoid the possibility of stock-out in the regular selling season. However, with a price premium for pre-orders, some experienced consumers will choose to wait until the regular selling season. As a result, both learning by the firm and the calculation of the stock-out probability will be much different and more involved.

A third issue concerns the assumption that the distribution of valuations is the same for experienced and inexperienced consumers. One straightforward generalization of our model is to assume that the distribution of valuations for experienced consumers is a rightward shift of that for inexperienced consumers (i.e., experienced consumers value the product more than inexperienced consumers on average). We expect many of the results in our paper to continue to hold in this extension. An alternative might be to assume a more general level of correlation between the two distribution functions. Much remains to be explored about this case.

Appendix

Explicit expressions for x_L and x_H:

Below we show that

$$\int_p^{+\infty} (v-p)f(v)\,\mathrm{d}v = (\mu - p)\overline{F}(p) + \sigma^2 f(p).$$

The explicit expressions for x_L and x_H will follow immediately from this equality. Applying the change of variable $z = (v-\mu)/\sigma$ to the integral on the left-hand side yields

$$\begin{aligned}
\int_p^{+\infty} (v-p)f(v)\,\mathrm{d}v &= \int_{\frac{p-\mu}{\sigma}}^{+\infty} (\mu + \sigma z - p)\phi(z)\,\mathrm{d}z \\
&= (\mu - p)\left(1 - \Phi\left(\frac{p-\mu}{\sigma}\right)\right) + \int_{\frac{p-\mu}{\sigma}}^{+\infty} \frac{\sigma z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\}\,\mathrm{d}z \\
&= (\mu - p)\overline{F}(p) + \frac{\sigma}{\sqrt{2\pi}} \int_{\frac{p-\mu}{\sigma}}^{+\infty} \exp\left\{-\frac{z^2}{2}\right\}\,\mathrm{d}\frac{z^2}{2}.
\end{aligned}$$

Applying the change of variable $u = z^2/2$ yields

$$
\begin{aligned}
(\mu - p)\overline{F}(p) &+ \frac{\sigma}{\sqrt{2\pi}} \int_{\frac{(p-\mu)^2}{2\sigma^2}}^{+\infty} \exp\{-u\}\ \mathrm{d}u \\
&= (\mu - p)\overline{F}(p) + \frac{\sigma}{\sqrt{2\pi}} \exp\left\{-\frac{(p-\mu)^2}{2\sigma^2}\right\} \\
&= (\mu - p)\overline{F}(p) + \sigma^2 f(p).
\end{aligned}
$$

Proof of Lemma 1:

It follows immediately from (1) and (2) that x_L and x_H increase in η, so we have part (iii) of Lemma 1.

(i) Below we show that

$$
x_H - x_L = \mu_H - \mu_L - (1-\eta)\left[\int_p^{+\infty} (v-p) f_H(v)\ \mathrm{d}v - \int_p^{+\infty} (v-p) f_L(v)\ \mathrm{d}v\right] > 0
$$

for all η and σ. Since $f_L(v)$ is a parallel shift of $f_H(v)$, the term in the square brackets can be rewritten as

$$
\begin{aligned}
&\int_p^{+\infty} (v-p) f_H(v)\ \mathrm{d}v - \int_{p+\mu_H-\mu_L}^{+\infty} (v - p - \mu_H + \mu_L) f_H(v)\ \mathrm{d}v \\
&\quad= \int_p^{p+\mu_H-\mu_L} (v-p) f_H(v)\ \mathrm{d}v + \int_{p+\mu_H-\mu_L}^{+\infty} (\mu_H - \mu_L) f_H(v)\ \mathrm{d}v \\
&\quad< \int_p^{p+\mu_H-\mu_L} (\mu_H - \mu_L) f_H(v)\ \mathrm{d}v + \int_{p+\mu_H-\mu_L}^{+\infty} (\mu_H - \mu_L) f_H(v)\ \mathrm{d}v \\
&\quad= (\mu_H - \mu_L) \int_p^{+\infty} f_H(v)\ \mathrm{d}v < \mu_H - \mu_L.
\end{aligned}
$$

Therefore,

$$
x_H - x_L > \mu_H - \mu_L - (1-\eta)(\mu_H - \mu_L) = \eta(\mu_H - \mu_L) \geq 0.
$$

(ii) In order to prove that x_L decreases in σ, we need to show that $\int_p^{+\infty} (v-p) f_L(v)\ \mathrm{d}v$ increases in σ.

$$
\begin{aligned}
\int_p^{+\infty} (v-p) f_L(v)\ \mathrm{d}v &= \int_p^{+\infty} v f_L(v)\ \mathrm{d}v - p \int_p^{+\infty} f_L(v)\ \mathrm{d}v \\
&= \int_{-\infty}^{+\infty} v f_L(v)\ \mathrm{d}v - \int_{-\infty}^{p} v\ \mathrm{d}F_L(v) - p(1 - F_L(p)) \\
&= \mu_L - \left(v F_L(v)\ |_{-\infty}^{p} - \int_{-\infty}^{p} F_L(v)\ \mathrm{d}v\right) - p + pF_L(p) \\
&= \mu_L + \int_{-\infty}^{p} F_L(v)\ \mathrm{d}v - p.
\end{aligned}
$$

Hence,

$$\frac{\partial}{\partial\sigma}\left(\int_{p}^{+\infty}(v-p)f_L(v)\,\mathrm{d}v\right)=\frac{\partial}{\partial\sigma}\left(\int_{-\infty}^{p}F_L(v)\,\mathrm{d}v\right).$$

Suppose $p<\mu_L$, then

$$\begin{aligned}\frac{\partial}{\partial\sigma}\left(\int_{-\infty}^{p}F_L(v)\,\mathrm{d}v\right)&=\frac{\partial}{\partial\sigma}\left(\sigma\int_{-\infty}^{\frac{p-\mu_L}{\sigma}}\Phi(z)\,\mathrm{d}z\right)\\&=\int_{-\infty}^{\frac{p-\mu_L}{\sigma}}\Phi(z)\,\mathrm{d}z-\frac{\sigma(p-\mu_L)}{\sigma^2}\Phi\left(\frac{p-\mu_L}{\sigma}\right)>0.\end{aligned}\tag{11}$$

Next, suppose $p>\mu_L$.

$$\int_{-\infty}^{p}F_L(v)\,\mathrm{d}v=\int_{-\infty}^{2\mu_L-p}F_L(v)\,\mathrm{d}v+\int_{2\mu_L-p}^{\mu_L}F_L(v)\,\mathrm{d}v+\int_{\mu_L}^{p}F_L(v)\,\mathrm{d}v.\tag{12}$$

Applying the change of variable $u=2\mu_L-v$ to the second integral yields

$$\int_{\mu_L}^{p}F_L(2\mu_L-u)\,\mathrm{d}u.$$

By symmetry of the normal distribution $F_L(v)+F_L(2\mu_L-v)=1$, thus the sum of the last two integrals in (12) equals $p-\mu_L$. Hence,

$$\frac{\partial}{\partial\sigma}\left(\int_{-\infty}^{p}F_L(v)\,\mathrm{d}v\right)=\frac{\partial}{\partial\sigma}\left(\int_{-\infty}^{2\mu_L-p}F_L(v)\,\mathrm{d}v\right).$$

The upper limit of the integration on the right-hand side is less than μ_L, hence, by (11) the above derivative is positive.

Thus, we have showed that x_L decreases in σ. The proof that x_H decreases in σ is similar.

Derivations of (4) and (5):[12]

As shown in Silver et al. (1998), the solution to the Newsvendor Problem (3) is q^* that satisfies

$$\Pr(D_2\le q^*)=\beta.\tag{13}$$

Moreover,

$$\pi(q^*)=(p-s)\int_0^{q^*}D_2g(D_2)\mathrm{d}D_2.$$

With $D_2\sim LN\left(\nu,\tau^2\right)$, (13) becomes

$$\Phi\left(\frac{\ln q^*-\nu}{\tau}\right)=\beta,$$

[12]The solution to the Newsvendor Problem under LN $\left(\nu,\tau^2\right)$ has been provided in lecture notes by Gallego (1995) without a proof.

or

$$q^* = \exp\{\nu + \tau z_\beta\}.$$

Next,

$$\pi(q^*) = (p-s)\int_0^{\exp\{\nu+\tau z_\beta\}} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(\ln D_2 - \nu)^2}{2\tau^2}\right\} \mathrm{d}D_2.$$

Applying the change of variable $u = \ln D_2$ yields

$$\begin{aligned}
\pi(q^*) &= (p-s)\int_{-\infty}^{\nu+\tau z_\beta} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(u-\nu)^2}{2\tau^2} + u\right\} \mathrm{d}u \\
&= (p-s)\exp\left\{\frac{(\nu+\tau^2)^2 - \nu^2}{2\tau^2}\right\}\int_{-\infty}^{\nu+\tau z_\beta} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(u-(\nu+\tau^2))^2}{2\tau^2}\right\} \mathrm{d}u \\
&= (p-s)\exp\left\{\nu + \frac{\tau^2}{2}\right\}\Phi(z_\beta - \tau) \\
&= (p-s)\left(1 - \Phi(\tau - z_\beta)\right)\exp\left\{\nu + \frac{\tau^2}{2}\right\}.
\end{aligned}$$

Derivation of (7):

Let $D_2 \sim \mathrm{LN}\left(\nu, \tau^2\right)$ and $q^* = \exp\{\nu + \tau z_\beta\}$. Then

$$\begin{aligned}
\eta &= \int_{q^*}^{+\infty} \frac{D_2 - q^*}{D_2} g(D_2)\,\mathrm{d}D_2 = 1 - G(q^*) - \int_{q^*}^{+\infty} \frac{q^*}{D_2} g(D_2)\,\mathrm{d}D_2 \\
&= 1 - G(\exp\{\nu+\tau z_\beta\}) - \int_{\exp\{\nu+\tau z_\beta\}}^{+\infty} \frac{\exp\{\nu+\tau z_\beta\}}{D_2}\frac{1}{D_2\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(\ln D_2 - \nu)^2}{2\tau^2}\right\} \mathrm{d}D_2.
\end{aligned}$$

Applying the change of variable $u = \ln D_2$ yields

$$\begin{aligned}
\eta &= 1 - \beta - \int_{\nu+\tau z_\beta}^{+\infty} \frac{\exp\{\nu+\tau z_\beta\}}{\exp\{u\}}\frac{1}{\exp\{u\}\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(u-\nu)^2}{2\tau^2}\right\}\exp\{u\}\,\mathrm{d}u \\
&= 1 - \beta - \exp\{\nu+\tau z_\beta\}\int_{\nu+\tau z_\beta}^{+\infty} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(u-\nu)^2}{2\tau^2} - u\right\} \mathrm{d}u \\
&= 1 - \beta - \exp\{\nu+\tau z_\beta\}\exp\left\{-\nu + \frac{\tau^2}{2}\right\}\int_{\nu+\tau z_\beta}^{+\infty} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\frac{(u-(\nu-\tau^2))^2}{2\tau^2}\right\} \mathrm{d}u \\
&= 1 - \beta - \exp\left\{\tau z_\beta + \frac{\tau^2}{2}\right\}\left(1 - \Phi\left(\frac{\nu+\tau z_\beta - (\nu-\tau^2)}{\tau}\right)\right) \\
&= 1 - \beta - \exp\left\{\tau z_\beta + \frac{\tau^2}{2}\right\}(1 - \Phi(z_\beta + \tau)).
\end{aligned}$$

Proof of Lemma 3:

The following properties of the standard normal distribution will be used in the proof:

$$\phi(u)\left(\frac{1}{u} - \frac{1}{u^3}\right) < 1 - \Phi(u) < \frac{\phi(u)}{u}, \quad u > 0 \tag{14}$$

and

$$\phi(u)\left(\frac{1}{|u|}-\frac{1}{|u|^3}\right) < \Phi(u) < \frac{\phi(u)}{|u|}, \quad u<0. \tag{15}$$

(i) First,

$$\eta(\beta,0) = 1-\beta-(1-\Phi(z_\beta)) = 1-\beta-(1-\beta) = 0.$$

Next,

$$\lim_{\tau_i\to+\infty}\eta(\beta,\tau_i) = 1-\beta-\lim_{\tau_i\to+\infty}\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}(1-\Phi(z_\beta+\tau_i)).$$

It follows from (14) that $1-\Phi(u)=\frac{\phi(u)}{u}(1+o(u^{-1}))$. Hence, the above expression can be rewritten as

$$\begin{aligned}\lim_{\tau_i\to+\infty}\eta(\beta,\tau_i) &= 1-\beta-\lim_{\tau_i\to+\infty}\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}\frac{\phi(z_\beta+\tau_i)}{z_\beta+\tau_i}\\ &= 1-\beta-\lim_{\tau_i\to+\infty}\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}\frac{\exp\left\{-\frac{(z_\beta+\tau_i)^2}{2}\right\}}{\sqrt{2\pi}(z_\beta+\tau_i)}\\ &= 1-\beta-\lim_{\tau_i\to+\infty}\frac{\exp\left\{-\frac{z_\beta^2}{2}\right\}}{\sqrt{2\pi}(z_\beta+\tau_i)} = 1-\beta.\end{aligned}$$

Finally,

$$\begin{aligned}\frac{\partial\eta}{\partial\tau_i} &= \frac{\partial}{\partial\tau_i}\left(-\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}(1-\Phi(z_\beta+\tau_i))\right)\\ &= -(z_\beta+\tau_i)\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}(1-\Phi(z_\beta+\tau_i))+\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}\phi(z_\beta+\tau_i)\\ &= (z_\beta+\tau_i)\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}\left[\frac{\phi(z_\beta+\tau_i)}{z_\beta+\tau_i}-(1-\Phi(z_\beta+\tau_i))\right].\end{aligned}$$

By (14) the term in the square brackets is positive, so $\partial\eta/\partial\tau_i > 0$.

(ii) First,

$$\eta(0,\tau_i) = 1-\lim_{z_\beta\to-\infty}\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}(1-\Phi(z_\beta+\tau_i)).$$

It follows from (15) that $\Phi(u)=\frac{\phi(u)}{|u|}(1+o(u^{-1}))$. Hence, the above expression can be rewritten as

$$\begin{aligned}\eta(0,\tau_i) &= 1-\lim_{z_\beta\to-\infty}\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}\left(1-\frac{\phi(z_\beta+\tau_i)}{|z_\beta+\tau_i|}\right)\\ &= 1-\lim_{z_\beta\to-\infty}\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}\left(1-\frac{\exp\left\{-\frac{(z_\beta+\tau_i)^2}{2}\right\}}{\sqrt{2\pi}|z_\beta+\tau_i|}\right)\\ &= 1-\lim_{z_\beta\to-\infty}\exp\left\{\tau_i z_\beta+\frac{\tau_i^2}{2}\right\}+\lim_{z_\beta\to-\infty}\frac{\exp\{-\frac{z_\beta^2}{2}\}}{\sqrt{2\pi}|z_\beta+\tau_i|} = 1.\end{aligned}$$

Next,

$$\eta(1,\tau_i) = -\lim_{z_\beta \to +\infty} \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\}(1-\Phi(z_\beta+\tau_i)).$$

Since $1-\Phi(u) = \frac{\phi(u)}{u}(1+o(u^{-1}))$,

$$\begin{aligned}\eta(1,\tau_i) &= -\lim_{z_\beta \to +\infty} \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\}\frac{\phi(z_\beta+\tau_i)}{z_\beta+\tau_i} \\ &= -\lim_{z_\beta \to +\infty} \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\}\frac{\exp\left\{-\frac{(z_\beta+\tau_i)^2}{2}\right\}}{\sqrt{2\pi}(z_\beta+\tau_i)} \\ &= -\lim_{z_\beta \to +\infty} \frac{\exp\left\{-\frac{z_\beta^2}{2}\right\}}{\sqrt{2\pi}(z_\beta+\tau_i)} = 0.\end{aligned}$$

Finally, we can write η as a function of z_β and τ_i,

$$\eta(z_\beta,\tau_i) = 1-\Phi(z_\beta) - \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\}(1-\Phi(z_\beta+\tau_i)).$$

Differentiating the above expression with respect to z_β yields

$$\begin{aligned}\frac{\partial\eta}{\partial z_\beta} &= -\phi(z_\beta) - \tau_i \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\}(1-\Phi(z_\beta+\tau_i)) + \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\}\phi(z_\beta+\tau_i) \\ &= -\tau_i \exp\left\{\tau_i z_\beta + \frac{\tau_i^2}{2}\right\}(1-\Phi(z_\beta+\tau_i)) < 0.\end{aligned}$$

Since z_β is increasing in β, it follows immediately that $\partial\eta/\partial\beta$ is also negative.

Derivatives of $\Pi^A(x_L)$, $\Pi^B(x_H)$, and $\Pi^C(p)$ with respect to α:

Rewriting the expressions (8) through (10) as functions of α yields

$$\Pi^A(x;\alpha) = \alpha m\left(\gamma\overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x-c) + (1-\alpha)m(x-c),$$

$$\Pi^B(x;\alpha) = \alpha m\left(\gamma\overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x-c) + \gamma(1-\alpha)m(x-c) + (1-\gamma)\pi_L,$$

and

$$\Pi^C(x;\alpha) = \alpha m\left(\gamma\overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x-c) + \gamma\pi_H + (1-\gamma)\pi_L,$$

where

$$\pi_L = (p-s)\left(1-\Phi(\tau_i - z_\beta)\right)(1-\alpha)m\overline{F}_L(p)$$

and

$$\pi_H = (p-s)\left(1-\Phi(\tau_i - z_\beta)\right)(1-\alpha)m\overline{F}_H(p).$$

Next, we differentiate $\Pi^A(x_L;\alpha)$ with respect to α,

$$\frac{\partial\Pi^A(x_L;\alpha)}{\partial\alpha} = m\left(\gamma\overline{F}_H(x_L) + (1-\gamma)\overline{F}_L(x_L)\right)(x_L-c) - m(x_L-c) < 0,$$

as

$$\gamma \overline{F}_H(x_L) + (1-\gamma)\overline{F}_L(x_L) < 1.$$

Differentiating $\Pi^B(x_H;\alpha)$ with respect to α yields

$$\frac{\partial \Pi^B(x_H;\alpha)}{\partial \alpha} = m\left(\gamma \overline{F}_H(x_H) + (1-\gamma)\overline{F}_L(x_H)\right)(x_H - c) - \gamma m(x_H - c).$$

The derivative can be positive or negative (we constructed examples). Finally,

$$\begin{aligned}
\frac{\partial \Pi^C(p;\alpha)}{\partial \alpha} &= m\left(\gamma \overline{F}_H(p) + (1-\gamma)\overline{F}_L(p)\right)\left((p-c) - (p-s)\left(1 - \Phi(\tau_i - z_\beta)\right)\right) \\
&= m\left(\gamma \overline{F}_H(p) + (1-\gamma)\overline{F}_L(p)\right)\left((p-s)\Phi(\tau_i - z_\beta) - (c-s)\right) \\
&> m\left(\gamma \overline{F}_H(p) + (1-\gamma)\overline{F}_L(p)\right)\left((p-s)\Phi(-z_\beta) - (c-s)\right)
\end{aligned}$$

Substituting $\Phi(-z_\beta) = 1 - \Phi(z_\beta) = 1 - \beta = (c-s)/(p-s)$ into the above inequality yields

$$\partial \Pi^C(p)/\partial \alpha > m\left(\gamma \overline{F}_H(p) + (1-\gamma)\overline{F}_L(p)\right)\left((c-s) - (c-s)\right) = 0.$$

Derivative of $\Pi^A(x_L)$ with respect to σ:

The expected profit function in region A, given by (8), depends on σ:

$$\begin{aligned}
\Pi^A(x;\sigma) &= m_e\left(\gamma \overline{F}_H(x) + (1-\gamma)\overline{F}_L(x)\right)(x-c) + m_i(x-c) \\
&= m_e\left(\gamma\left(1 - \Phi\left(\frac{x-\mu_H}{\sigma}\right)\right) + (1-\gamma)\left(1 - \Phi\left(\frac{x-\mu_L}{\sigma}\right)\right)\right)(x-c) + m_i(x-c).
\end{aligned}$$

For any x in region A, $x < x_L < \mu_L$,

$$\frac{\partial}{\partial \sigma}\left(1 - \Phi\left(\frac{x-\mu_H}{\sigma}\right)\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu_H)^2}{2\sigma^2}\right\} \frac{x-\mu_H}{\sigma^2} < 0$$

and

$$\frac{\partial}{\partial \sigma}\left(1 - \Phi\left(\frac{x-\mu_L}{\sigma}\right)\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu_L)^2}{2\sigma^2}\right\} \frac{x-\mu_L}{\sigma^2} < 0.$$

It follows that

$$\frac{\mathrm{d}\Pi^A(x_L;\sigma)}{\mathrm{d}\sigma} = \frac{\partial \Pi^A(x_L;\sigma)}{\partial \sigma} + \frac{\partial \Pi^A(x_L;\sigma)}{\partial x_L}\frac{\partial x_L}{\partial \sigma} < 0,$$

because the first term is negative as was shown above, and the second term is negative because $\partial \Pi^A(x_L;\sigma)/\partial x > 0$ by Assumption 1 and $\partial x_L/\partial \sigma < 0$ by Lemma 1(ii).

References

[1] Cachon, Gérard P., 2004, "The Allocation of Inventory Risk in a Supply Chain: Push, Pull, and Advance-Purchase Discount Contracts," *Management Science*, 50(2), pp. 222-238.

[2] Chen, Hairong, and Mahmut Parlar, 2005, "Dynamic Analysis of the Newsboy Model with Early Purchase Commitments," *International Journal of Services and Operations Management*, 1(1), pp. 56-74.

[3] Chu, Leon Yang and Hao Zhang, 2011, "Optimal Preorder Strategy with Endogenous Information Control," *Management Science*, 57(6), pp. 1055-1077.

[4] Gallego, Guillermo, 1995, "Production Management," *lecture notes*, Columbia University, New York, NY.

[5] Liu, Qian, and Garrett J. van Ryzin, 2008, "Strategic Capacity Rationing to Induce Early Purchases," *Management Science*, 54(6), pp. 1115-1131.

[6] McCardle, Kevin, Kumar Rajaram, and Christopher S. Tang, 2004, "Advance Booking Discount Programs under Retail Competition," *Management Science*, 50(5), pp. 701-708.

[7] Prasad, Ashutosh, Kathryn E. Stecke, and Xuying Zhao, 2011, "Advance Selling by a Newsvendor Retailer," *Production and Operations Management*, 20(1), pp. 129-142.

[8] Silver, Edward A., David F. Pyke, and Rein Peterson, 1998, *Inventory Management and Production Planning and Scheduling*, 3rd edition, New York: Willey.

[9] Tang, Christopher S., Kumar Rajaram, Aydin Alptekinoğlu, and Tihong Ou, 2004, "The Benefits of Advance Booking Programs: Modal and Analysis," *Management Science*, 50(4), pp. 465-478.

[10] Taylor, Terry A., 2006, "Sale Timing in a Supply Chain: When to Sell to the Retailer," *Manufacturing & Service Operations Management*, 8(1), pp.23-42.

[11] Weng, Z. Kevin, and Mahmut Parlar, 1999, "Integrating Early Sales with Production Decisions: Analysis and Insights," *IIE Transactions*, 31(11), pp. 1051-1060.

[12] Xie, Jinhong, and Steven M. Shugan, 2001, "Electronic Tickets, Smart Cards, and Online Prepayments: When and How to Advance Sell," *Marketing Science*, 20(3), pp. 219-243.

[13] Zhao, Xuying, and Kathryn E. Stecke, 2010, "Pre-Orders for New To-be-Released Products Considering Consumer Risk Aversion," *Production and Operations Management*, 19(2), pp. 198-215.